Lecture 16

Plan: We did lots of preparation for Laplace transform. Today we use it to solve D.E.: §7.5 solving I.V.Ps

Recap: • §7.2:

$f(t)$	$\mathcal{L}\{f\}(s)$	Region of existence
c (a constant)	$\frac{c}{s}$	$s>0$
t^n, $n>0$ integer	$\frac{n!}{s^{n+1}}$	$s>0$
e^{kt}, $k\in\mathbb{R}$	$\frac{1}{s-k}$	$s>k$
$\sin bt$, $b\in\mathbb{R}$	$\frac{b}{s^2+b^2}$	$s>0$
$\cos bt$, $b\in\mathbb{R}$	$\frac{s}{s^2+b^2}$	$s>0$
$e^{at}\sin(bt)$, $a,b\in\mathbb{R}$	$\frac{b}{(s-a)^2+b^2}$	$s>a$
$e^{at}\cos(bt)$, $a,b\in\mathbb{R}$	$\frac{s-a}{(s-a)^2+b^2}$	$s>a$

§7.3.

Thm 1 $\mathcal{L}\{e^{\beta t} f(t)\}(s) = \mathcal{L}\{f\}(s-\beta)$

Thm 2 $\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0)$

Thm 3 $\mathcal{L}\{\int_0^t f(x)dx\}(s) = \frac{1}{s}\mathcal{L}\{f(t)\}(s)$

Thm 4

$$\mathcal{L}\{f^{(n)}\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$$

In particular, $n=2 \Rightarrow$

$$\mathcal{L}\{f^{(2)}\}(s) = s^2\mathcal{L}\{f\}(s) - sf(0) - f'(0)$$

Thm 5: $\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n F}{ds^n}(s)$,

$n \in \mathbb{Z}$, $n \geq 1$

Ideas of solving D.E by using Laplace transform:

Reduce the D.E to a simpler algebraic/arithmetic eqn.

This idea has been used in the past

$$ay'' + by' + cy = 0 \quad (1)$$

$$a\lambda^2 + b\lambda + c = 0 \quad ①$$

① is a lot easier than (1).

← Cauchy-Euler eqn

$$ax^2y'' + bxy' + cy = 0 \quad (2)$$

$$a\lambda^2 + (b-a)\lambda + c = 0 \quad ②$$

② is a lot easier than (2).

Two examples:

E.g 1: Use Laplace transform to solve I.V.P:

$$\begin{cases} y''-2y'+5y=-8e^{-t} & (3) \\ y(0)=2, \ y'(0)=12 \end{cases}$$

A: Step 1: Apply Laplace transform $\mathcal{L}$ to the D.E.

Apply $\mathcal{L}$ to both sides of (3)

$$\mathcal{L}\{y''-2y'+5y\} = \mathcal{L}\{-8e^{-t}\}$$

$$\Rightarrow \quad \mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + 5\mathcal{L}\{y\} = -8\mathcal{L}\{e^{-t}\} \qquad (*)$$

Write $Y(s) = \mathcal{L}\{y\}(s)$.

Compute:

$$\mathcal{L}\{y''\} = s^2\mathcal{L}\{y\}(s) - sy(0) - y'(0)$$

Thm 4 §7.3

$$= s^2 Y - s\cdot 2 - 12;$$

$$\mathcal{L}\{y'\} = s\mathcal{L}\{y\}(s) - y(0)$$

Thm 2 §7.3

$$= sY - 2$$

Recall for $s>k$

$\mathcal{L}\{e^{kt}\}(s) = \frac{1}{s-k}$

let $k=-1 \Rightarrow$

$\mathcal{L}\{e^{-t}\}(s) = \frac{1}{s+1}$

$$\mathcal{L}\{e^{-t}\} = \frac{1}{s+1}$$

Thus $(*)$ becomes:

$$s^2Y - 2s - 12 - 2(sY - 2) + 5Y = -8\frac{1}{s+1}$$

This eqn is much easier to solve than the D.E (3)

$$\Rightarrow (s^2-2s+5)Y - 2s - 8 = \frac{-8}{s+1}$$

$$\Rightarrow (s^2-2s+5)Y = \frac{-8}{s+1} + 2s + 8$$

$$= \frac{2s^2+10s}{s+1}$$

$$\Rightarrow$$

$$\mathcal{L}\{y\} = Y = \frac{2s^2+10s}{(s^2-2s+5)(s+1)}$$

Q: what is y?

A: $y = \mathcal{L}^{-1}\{Y\}$!

$$\Rightarrow y = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{2s^2+10s}{(s^2-2s+5)(s+1)}\right\}$$

The RHS is already computed in

Lecture 15 E.g 2 and E.g 4

$$y = \mathcal{L}^{-1}\left\{\frac{2s^2+10s}{(s^2-2s+5)(s+1)}\right\}$$

$$= 3e^{t}\cos(2t) + 4e^{t}\sin(2t) - e^{-t}$$

Remark: In the above E.g 1, one can also use the old method:

step 1: Solve the associated homogeneous eqn

$$y'' - 2y' + 5y = 0$$

Find two L.I. solns: y_1, y_2.

Step 2: Use undetermined coefficients method or variation of parameters.

We leave this as an exercise.

However, the following E.g 2 cannot be solved by the old way, and can be solved by using Laplace transform.

Before doing E.g 2, we first introduce

proposition:

If f is piecewise continuous on $[0, \infty)$ and of exponential order α for some $\alpha \in \mathbb{R}$, then

$$\lim_{s \to \infty} \mathcal{L}\{f\}(s) = \lim_{s \to \infty} \int_0^\infty e^{-st} f(t)\,dt = 0$$

Rough Pf: Since f is of exponential order α,

$\Rightarrow$ there exist T, M s.t

$$0 \leq |f(t)| \leq M e^{\alpha t} \text{ for } t \geq T$$

$$\Rightarrow \quad 0 \leq e^{-st} |f(t)| \leq M e^{-st} e^{\alpha t} \text{ for } t \geq T$$

$$\Rightarrow \quad 0 \leq \left| \int_T^\infty e^{-st} f(t)\,dt \right| \leq \underbrace{\int_T^\infty M e^{-st} e^{\alpha t}\,dt}_{\downarrow\, 0 \text{ E.x}}$$

Let $s \to \infty$: $\quad 0 \to 0, \quad \to 0, \quad \to 0$ E.x

$$\Rightarrow \quad \lim_{s \to \infty} \int_T^\infty e^{-st} f(t)\,dt = 0$$

Note

$$\int_0^\infty e^{-st} f(t)\,dt = \int_0^T e^{-st} f(t)\,dt + \int_T^\infty e^{-st} f(t)\,dt$$

Hint: $|f(t)| \le N$ on $[0,T]$ for some $N>0$.

$\downarrow$ E.x $\qquad\qquad \downarrow$

$0 \qquad\qquad 0$

Remark: When we use Laplace transform to solve D.E, we normally require

$$\lim_{s\to\infty} Y(s) = \lim_{s\to\infty} \mathcal{L}\{y\}(s) = 0$$

Free information you can use.

E.g 2: Solve the I.V.P

$$\begin{cases} y'' + 2ty' - 4y = 1 & (4) \\ y(0) = y'(0) = 0. \end{cases}$$

A: Apply $\mathcal{L}$ to (4) $\Rightarrow$

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{ty'\} - 4\mathcal{L}\{y\} = \mathcal{L}\{1\}.$$

Compute:

write $Y(s) = \mathcal{L}\{y\}(s)$.

Thm 4 §7.3

$$\mathcal{L}\{y''\}(s) = s^2 Y - s y(0) - y'(0)$$
$$= s^2 Y.$$

Q: How to compute $\mathcal{L}\{t y'\}(s)$?

Warning: $\mathcal{L}\{t y'\} \neq t \mathcal{L}\{y'\}$!.

Note t is NOT a constant.

Use Thm 5 in §7.3:

Thm 5 in §7.3:

write $F(s) = \mathcal{L}\{f\}(s)$

$\Rightarrow$

$$\mathcal{L}\{t f\}(s) = (-1) \frac{dF}{ds}$$

By Thm 2 in §7.3. $\Rightarrow$

$$\mathcal{L}\{y'\}(s) = s \mathcal{L}\{y\} - y(0)$$
$$= s Y(s) - y(0)$$

$$\mathcal{L}\{t y'\}(s) = (-1) \frac{d}{ds}\left(\mathcal{L}\{y'\}\right)$$
$$= (-1)\left(sY - y(0)\right)'$$
$$= (-1)(sY)'$$
$$= (-1)(sY' + Y)$$
$$= -sY' - Y.$$

$$\mathcal{L}\{1\}(s) = \frac{1}{s}.$$

Table §7.2

$$\Rightarrow \quad s^2 Y + 2(-sY' - Y) - 4Y = \frac{1}{s}$$

$$\Rightarrow \quad -2sY' + (s^2-6)Y = \frac{1}{s}$$

$$\Rightarrow \quad Y' + \underbrace{\left(-\frac{s}{2} + \frac{3}{s}\right)}_{P(s)} Y = \underbrace{\frac{-1}{2s^2}}_{Q(s)} \qquad (5)$$

Q: How to solve (5)?

A: Use Integrating factor!

§2.3

To solve

$$\frac{dy}{dx} + P(x)y = Q(x)$$

step 1. compute

$$\mu(x) = e^{\int P(x)dx}$$

Step 2: Multiple the D.E by μ

$$\mu\frac{dy}{dx} + \mu P(x)y = \mu Q(x)$$

$$\Rightarrow \quad \frac{d}{dx}(\mu y) = \mu Q(x)$$

$$\Rightarrow \quad \mu y = \int \mu Q(x)dx$$

$$\Rightarrow \quad y = \frac{1}{\mu}\int \mu Q(x)dx$$

Step 1: compute

$$\mu(s) = e^{\int\left(-\frac{s}{2}+\frac{3}{s}\right)ds}$$

$$= s^3 e^{-s^2/4}$$

E.X

☆ Hint: when you use Laplace transform to solve D.E, you can assume $s>0$!

Step 2:

$$Y(s) = \frac{1}{\mu(s)}\int \mu(s)Q(s)\,ds$$

$$= s^{-3}e^{s^2/4}\int s^3 e^{-s^2/4}\left(\frac{-1}{2s^2}\right)ds$$

$$= s^{-3}e^{s^2/4}\int\left(\frac{-s}{2}\right)e^{-s^2/4}ds$$

$$= s^{-3}e^{s^2/4}\left(e^{-s^2/4} + C\right)$$

$\Longrightarrow$

$$Y(s) = \frac{1}{s^3} + C\frac{e^{s^2/4}}{s^3}, \quad C \text{ constant}.$$

Q: Can we determine C?

A: Yes! Use the "free information":

$$\lim_{s\to\infty} Y(s) = 0.$$

In our case,

$$\lim_{s\to\infty} Y(s) = \lim_{s\to\infty}\Big(\underbrace{\frac{1}{s^3}}_{\to 0} + C\underbrace{\frac{e^{s^2/4}}{s^3}}_{\to\infty}\Big)$$

Let's compute $\displaystyle\lim_{s\to\infty}\frac{e^{s^2/4}}{s^3}$ ← L'Hopital

$$\lim_{s\to\infty}\frac{e^{s^2/4}}{s^3} \overset{\text{L'H}}{=} \lim_{s\to\infty}\frac{e^{s^2/4}\,\frac{2s}{4}}{3s^2} = \lim_{s\to\infty}\frac{1}{6}\,\frac{e^{s^2/4}}{s}$$

$$\overset{\text{L'H}}{=} \lim_{s\to\infty}\frac{1}{6}\,\frac{e^{s^2/4}\,\frac{2s}{4}}{1}$$

$$= \lim_{s\to\infty}\frac{1}{12}\,s\,e^{s^2/4} = \infty$$

↑ means "$+\infty$"

Hence

$$\lim_{s\to\infty}\left(\frac{1}{s^3} + c\frac{e^{s^2/4}}{s^3}\right) = \begin{cases} \infty & c>0 \\ 0 & c=0 \\ -\infty & c<0 \end{cases}$$

Hence $\lim_{s\to\infty} Y(s) = 0 \Rightarrow c=0$

Hence

$$\mathcal{L}\{y\} = Y(s) = \frac{1}{s^3}$$

$$\Longrightarrow y(t) = \mathcal{L}^{-1}\{Y\}(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}(t)$$

$$= \frac{t^2}{2}.$$

Recall the table in §7.2

$$\mathcal{L}\{t^n\}(s) = \frac{n!}{s^{n+1}}$$

Let $n=2 \Rightarrow$

$$\mathcal{L}\{t^2\}(s) = \frac{2}{s^3}$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\}(t) = t^2$$

$$\Rightarrow 2\cdot\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}(t) = t^2$$

Hence the soln to the I.V.P.:

$$y(t) = \frac{t^2}{2}.$$